Numerical Methods Cheat Sheet

This topic offers methods to estimate several problems that you may not be able to solve analytically You will be able to find numerical solutions for first-order differential equations and extend one of the methods to find numerical solutions of second order differential equations. You will also be able to approximate definite integrals using Simpson's rule. These methods are relevant in computing- computers often use these methods to find solutions to problems rather than solving it analytically.

Solving first-order differential equations

Methods to solve first-order differential equations of the form $\frac{dy}{dx} = f(x, y)$ have been explored previously, but in some cases it might be difficult or impossible to solve using an analytic method. When using an analytical method, you find a general solution, the use the conditions given in the question to find the particular solution. The general solution corresponds to an infinite set of curves (which can be illustrated by a tangent field or compass point diagram by considering the gradients) and the particular solution narrows this down to one particular curve.

By considering the gradients, we can solve differential equations iteratively- this is especially useful for equations that cannot be solved analytically.

- Euler's method for approximating solutions to first-order differential equations is given by

$$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$$

Which can also be written as an iterative formula

$$y_{r+1} \approx y_r + h\left(\frac{dy}{dx}\right)_r, \qquad r = 0,1,2,\ldots$$

This finds the y-value of the differential equation given an initial condition, which can then be used to find the gradient at this initial point, denoted $\left(\frac{dy}{dx}\right)_0$. We then approximate the next point on the solution curve by moving a small amount along the tangent line, this small line is denoted h, and finding gradient at this new point.

Example 1: Use Euler's method to estimate the value at $x = 2$ of the particular solution to the differential equation $\frac{dy}{dx} = \sqrt{3e^x + 4e^y}$ using two iterations, which passes through the point (1,2).

Work out the step size h.	As we start at $x = 1$, and need to estimate the value at $x = 2$ in two steps, the step size will be $h = 0.5$
Calculate $\left(\frac{dy}{dx}\right)_0$.	$\left(\frac{dy}{dx}\right)_0 = \sqrt{3e + 4e^2}$
Calculate y_1 using the iterative formula.	$y_{r+1} \approx y_r + h\left(\frac{dy}{dx}\right)_r$ $y_1 = 2 + 0.5(\sqrt{3e + 4e^2})$
Calculate $\left(\frac{dy}{dx}\right)_1$.	Using the step size, we know that $x_1 = x_0 + h = 1.5$, and we have calculated that $y_1 = 2 + 0.5(\sqrt{3e + 4e^2})$. $\left(\frac{dy}{dx}\right)_1 = \sqrt{3e^{1.5} + 4e^{2+0.5(\sqrt{3e+4e^2})}}$
Calculate y_2 using the iterative formula.	$y_2 = y_1 + h\left(\frac{dy}{dx}\right)_0$ $y_2 = 2 + 0.5\left(\sqrt{3e + 4e^2}\right) + 0.5\left(\sqrt{3e^{1.5} + 4e^{2+0.5(\sqrt{3e+4e^2})}}\right)$ $y_2 = 17.822$ (to 3dp)

This is a method of estimation, and thus we can change some parameters to make the estimation more accurate. To make Euler's method more accurate, we need to **reduce the step length** (denoted h).

However, there are alternative methods that we can use to make our estimations more accurate:

- The midpoint method for approximating solutions to first-order differential equations uses the formula

$$\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - y_{-1}}{2h}$$

Which can be written iteratively as

$$y_{r+1} \approx y_{r-1} + 2h\left(\frac{dy}{dx}\right)_r, r = 0,1,2,\ldots$$

Example 2: Use the midpoint formula to estimate the value at $x = 2.75$ of the particular solution to the differential equation $\frac{dy}{dx} = x^2 + y^3$, which passes through the point (2,3), using a step length of 0.25.

Write down the information you know and compare to the information that you need.	$x_0 = 2, y_0 = 3, h = 0.25$ $x_1 = 2.25$ $x_2 = 2.5$ $\left(\frac{dy}{dx}\right)_0 = 2^2 + 3^3 = 31$
Notice that with the midpoint formula, the smallest value of the index r that we can calculate y_{r+1} for is $r = 1$, thus we will be calculating y_2. To do this, we need to calculate $\left(\frac{dy}{dx}\right)_1$, but we can't do this without y_1, so we must use Euler's formula to find y_1.	$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$ $y_1 = 2 + 0.25(31) = 9.75$
Calculate $\left(\frac{dy}{dx}\right)_1$ by substituting x_1 and y_1 into $\frac{dy}{dx}$.	$\left(\frac{dy}{dx}\right)_1 = 2.25^2 + 9.75^2 = 100.125$
Find y_2 by using the midpoint formula.	$y_2 = y_0 + 2h\left(\frac{dy}{dx}\right)_1$ $y_2 = 2 + 2(0.25)(100.125)$ $y_2 = 52.0625$
Find $\left(\frac{dy}{dx}\right)_2$.	$\left(\frac{dy}{dx}\right)_2 = 2.5^2 + 52.0625^3$ $= 141121.8596$
Calculate y_3 using the midpoint method.	$y_3 = y_1 + 2h\left(\frac{dy}{dx}\right)_2$ $= 9.75 + 2(0.25)(\,141121.8596)$ $= 70570.67981 \approx 70570.680$ (to 3 d.p.)

Solving second-order differential equations

Euler's method can be extended to find approximate solutions to second-order differential equations of the form $\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$

- Euler's method for approximating solutions to second-order differential equations is given by

$$\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

This can also be written iteratively as

$$y_{r+1} \approx 2y_r - y_{r-1} + h^2\left(\frac{d^2y}{dx^2}\right)_r, \qquad r = 0,1,2,\ldots$$

Once again, you may need to use Euler's formula to find missing values if necessary. It is also important to pay close attention to the indexes of y and $\frac{d^2y}{dx^2}$, it is very easy to get them mixed up!

Example 3: For the second-order differential equation $\frac{d^2y}{dx^2} = x^2 - y^2$. When $x = 0, y = 1$ and $\frac{dy}{dx} = 2$. Use the approximations $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$ and $\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$ to obtain estimates for y at $x = 0.2$ and $x = 0.4$.

Write down the information that you know and compare to the information that you need- we need to find y_1 and y_2.	$x_0 = 0, y_0 = 1, \left(\frac{dy}{dx}\right)_0 = 2, h = 0.2$ $x_1 = 0.2, x_2 = 0.4$
Two values of y are needed to substitute into the equation- y_1 can be found by using Euler's formula for first-order differential equations.	$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$ $= 1 + (0.2)(2)$ $= 1.4$
Find $\left(\frac{d^2y}{dx^2}\right)_1$.	$\left(\frac{d^2y}{dx^2}\right)_1 = (0.2)^2 - 1.4^2$ $= -1.92$
Find y_2 using the iterative formula.	$y_2 = 2y_1 - y_0 + h^2\left(\frac{d^2y}{dx^2}\right)_1$ $= 2(1.4) - 1 + 0.2^2(-1.92)$ $= 1.7232$

Example 4: The curve $y = f(x)$ satisfies the differential equation $\frac{d^2y}{dx^2} = \sin x + y^2 + \frac{dy}{dx}$. When $x = \frac{\pi}{2}, y = 1$ and $\frac{dy}{dx} = 3$.

Use the approximations $\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_0}{h}$ and $\left(\frac{d^2y}{dx^2}\right)_0 \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2}$, with $h = \frac{\pi}{8}$ to estimate the value of y when $x = \frac{5\pi}{8}$.

Write down the information that you know and compare to the information that you need- we need to find y_1.	$x_0 = \frac{\pi}{2}, y_0 = 1, \left(\frac{dy}{dx}\right)_0 = 3, h = \frac{\pi}{4}$ $x_1 = \frac{5\pi}{8}$
Find $\left(\frac{d^2y}{dx^2}\right)_0$ using the initial conditions.	$\left(\frac{d^2y}{dx^2}\right)_0 = \sin\left(\frac{\pi}{4}\right) + 1^2 + 3$ $= \frac{8 + \sqrt{2}}{2} = 4.7071\ldots$
Use the approximations for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to form simultaneous equations in y_1 and y_{-1}.	$\left(\frac{dy}{dx}\right)_0 \approx \frac{y_1 - y_{-1}}{2h}$ $3 = \frac{y_1 - y_{-1}}{\pi \div 4} \Rightarrow y_1 - y_{-1} = \frac{3\pi}{4}$ $\left(\frac{d^2y}{dx^2}\right)_0 = \frac{y_1 - 2y_0 + y_{-1}}{h^2}$ $\frac{8 + \sqrt{2}}{2} = \frac{y_1 - 2 + y_{-1}}{(\frac{\pi}{8})^2} \Rightarrow 2.725895 = y_1 + y_{-1}$
Solve the equations simultaneously.	$y_1 - y_{-1} = \frac{3\pi}{4}$ $y_1 + y_{-1} = 2.725895$ $2y_1 = 5.08209$ $y_1 = 2.5410$

Simpson's rule

Simpson's rule is a way of estimating the value of a definite integral of the form $I = \int_a^b f(x)dx$. If you consider the curve $f(x)$, the integral is the area under this curve. Simpson's rule splits the curve up into sections, but unlike the trapezium rule, instead of approximating the sections of the curve by a straight line, the sections are paired off and a quadratic curve approximates each curve- because of this, Simpson's rule only works for an even number of strips

- Simpson's rule for $2n$ strips of width h is given by:

$$\int_a^b f(x)dx \approx \frac{1}{3}h(y_0 + 4(y_1 + y_3 + \cdots + y_{2n-1}) + 2(y_2 + y_4 + \cdots + y_{2n-2}) + y_{2n})$$

This formula is not given in the formula book so will need to be learned- it can be more easily remembered by the informal definition:

$$\int_a^b f(x)dx \approx \frac{1}{3}h(\text{sum of end points}) + 4(\text{sum of odd values}) + 2(\text{sum of even values}))$$

When attempting exam questions with Simpson's rule, the best plan is to construct a table of the x and y-values.

Example 5: Use Simpson's rule with 4 intervals to estimate $\int_1^2 \sqrt{x^3 + 2}\, dx$

Calculate the step length.	We are using 4 steps to evaluate the integral between 1 and 2. Thus, $h = 0.25$
Construct a table with the x and y values, the y-values are calculated by substituting the x values into the original equation.	x_i: 1, 1.25, 1.5, 1.75, 2 y_i: $\sqrt{3}$, 7.9529, 4.6368, 10.8513, $\sqrt{10}$
Substitute the values into the equation.	$\int_1^2 \sqrt{x^3 + 2}\, dx \approx \frac{1}{3}(0.25)((\sqrt{3} + \sqrt{10})$ $+ 4(7.9529 + 10.8513)$ $+ 2(4.6368))$ $= 2.3613$

If you have a graphics calculator and it is permitted in your exams it is often a good idea to check your estimate using the integral function- often they will not be identical, especially with 'wide' sections, or to a large number of decimal places, but they should be similar.